1/16/26 No class Monday. Quiz Wednesday!

Last time:

Ex (again from book §1.2)

$$\left| \begin{array}{l} x_1 - x_2 + 4x_5 = 2 \\ x_3 - x_5 = 2 \\ x_4 - x_5 = 3 \end{array} \right|$$

why is this a convenient form for our equations to be in?

(i) Solve for the leading variables:

$$x_1 = 2 + x_2 - 4x_5$$
$$x_3 = 2 + x_5$$
$$x_4 = 3 + x_5$$

(ii) Pick arbitrary values for x_2 and x_5:

$$x_2 = r \qquad \text{(e.g. } x_2 = 87.6$$
$$x_5 = t \qquad x_5 = 114.66 \text{)}$$

(iii) General solution:

$$x_1 = 2 + r - 4t$$
$$x_2 = r$$
$$x_3 = 2 + t$$
$$x_4 = 3 + t$$
$$x_5 = t$$

- there are infinitely many solutions since r and t can be anything.
- 2-dim'l solution set

Matrix point of view:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

(reduced row-echelon form)
(RREF)

This is why RREF is so convenient! We were able to get the general solution because we had a matrix in RREF.

More precisely:

Reduced row echelon form (RREF)

P1 the 1^{st} nonzero entry in each row is 1 ("leading 1")

P2 for any such circled "1", the rest of that column is 0

P3 the circled 1's go left to right as you go down the rows

More precisely: if a row contains a leading 1 then each row above it contains a leading 1 further to the left.

Then you can assign random values to the variables not corresponding to a circled ("leading") 1 and solve for the variables that do correspond to leading 1's.

The leading 1's are also called pivots.

In an <u>augmented</u> matrix, leading 1's are <u>never</u> in the last column (by def'n).
- If $[0 \cdots 0 \mid 1]$ happens to be a row (<u>see below</u>), that 1 isn't "leading"

The operations we're allowed to perform on the equations/rows are called
<u>elementary row operations</u>:

- divide a row by a nonzero scalar (ie nonzero number)
- add / subtract a multiple of a row from another row
- swap 2 rows

The process of solving a linear system in this way is called
Gauss-Jordan elimination

<u>Example</u> (from book)

$$\left|\begin{array}{l} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 = 2 \\ x_1 + 2x_2 - x_3 + 2x_4 = 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9 \end{array}\right|$$

1st convert to augmented matrix

$$\left[\begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \div 2$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \begin{array}{l} \\ -(\mathrm{I}) \\ -3(\mathrm{I}) \\ -5(\mathrm{I}) \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right] \begin{array}{l} -(\mathrm{II}) \\ \\ +2(\mathrm{II}) \\ \\ \end{array}$$

go row by row first

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right] \begin{array}{l} +(\mathrm{III}) \\ \\ \\ -(\mathrm{III}) \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (\mathrm{II}) \leftrightarrow (\mathrm{III})$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

$x_1 = 2 - 2x_2 - 3x_5$ pick an arbitrary value r for x_2

$x_3 = 4 + x_5$ " " " " t for x_5

$x_4 = 3 + 2x_5$

gen'l sol'n

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 2r - 3t \\ r \\ 4 + t \\ 3 + 2t \\ t \end{bmatrix}$$

<u>Note</u> – if you ever get

$$0 \quad 0 \quad \cdots \quad 0 \;\Big|\; \neq 0$$

then there's no solution!

<u>Ex</u> $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right]$ the eqns are $\left|\begin{array}{l} x_1 + 2x_2 = 3 \\ x_3 = 2 \\ 0 = 4 \end{array}\right|$

On the other hand

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longleftrightarrow \begin{array}{l} x_1 + 2x_2 = 3 \\ x_3 = 2 \end{array} \longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 2t \\ t \\ 2 \end{bmatrix}$$

gen'l soln

Let's bring some things together.

Def A system of lin eqns is consistent if there's at least one sol'n

A linear system is inconsistent if the RREF includes the row $[0 \ \cdots \ 0 \mid 1]$

Note If a lin. system is consistent then it either has a unique solution or infinitely many. (Recall geometry from Monday.)

If it's inconsistent, it has no sol'n.

Note that the steps in Gauss-Jordan elimination work even if A is just a matrix, not necess. an augmented matrix.

We'll use this fact.

Def If A is a matrix, then RREF(A) is the corresp. matrix in RREF after performing Gauss-Jordan elimination

Def If A is any matrix then the rank of A, denoted rank(A), is the number of leading 1's in RREF(A).